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Angular Rate Estimation Using Fixed and Vibrating Triaxial Acceleration Measurements

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Nomenclature

$\boldsymbol{a}_{a/b}$	=	acceleration of point a with respect to reference frame b
F_j	=	jth fixed sensor
L	=	body frame component of sensor system geometry, m
n	=	vibration amplitude of vibrating sensors, m
p	=	$\boldsymbol{I}_s$ component of angular velocity in the sensor reference frame, rad/s
q	=	$\boldsymbol{J}_s$ component of angular velocity in the sensor reference frame, rad/s
r	=	$\boldsymbol{K}_s$ component of angular velocity in the sensor reference frame, rad/s
$\boldsymbol{r}_{a\to b}$	=	position vector from point a to point b
V_j	=	jth vibrating sensor
$\boldsymbol{v}_{a/b}$	=	velocity of point a with respect to reference frame b
$\alpha_{a/b}$	=	angular acceleration vector of body a with respect to reference frame b
Δx	=	$\boldsymbol{I}_s$ component of change in position vector in sensor frame, m
Δy	=	$\boldsymbol{J}_s$ component of change in position vector in sensor frame, m
Δz	=	$\boldsymbol{K}_s$ component of change in position vector in sensor frame, m
ω	=	circular frequency of vibrating sensors, rad/s
$\boldsymbol{\omega}_{a/b}$	=	angular velocity vector of body a with respect to reference frame b

Subscripts

x	=	$\boldsymbol{I}_s$ axis component
y	=	$\boldsymbol{J}_s$ axis component
z	=	$\boldsymbol{K}_s$ axis component

Introduction

THE introduction of a wide variety of microelectromechanical systems into the marketplace has opened the door to incorporate small and relatively inexpensive sensors into air vehicles in new and innovative ways. A case in point is small- and medium-caliber smart projectiles. Until recently, sensor size, durability, and cost issues have prevented active control of gun-launched projectiles. Whereas the development of new MEMS sensors is a very active area of inquiry with new devices entering the marketplace regularly, the most highly developed motion sensor for use on smart weapons is the accelerometer. Accelerometers are particularly attractive for gun-launched projectiles and rockets because they are rugged and can survive high-acceleration levels typical at launch.

A number of investigators have examined the use of linear acceleration measurements to compute angular rates and angular accelerations of a body.[1,2] A link between linear acceleration, angular velocity, and angular acceleration is provided by rigid-body kinematics.[3] Clusters of triaxial accelerometers have been proposed to compute body angular rates. By exploiting properties of skew symmetric matrices, Angeles[4] established a method to compute angular velocity and angular acceleration of a body using acceleration measurements. Nusholtz[5] used spherical geometric analysis to perform similar computations. Costello and Jitpraphai[6] obtained a closed-form solution for angular rates using triaxial acceleration measurements and also showed that it is not possible to determine the sign of the angular rates purely using rigid-body kinematic relationships. The problem of determining the algebraic sign of angular rates can be avoided by using acceleration sensors that are fixed to the body along with acceleration sensors that vibrate in a known manner with respect to the body. Merhav[7] developed a method to estimate body angular rates using three single-axis, fixed and vibrating acceleration measurements. This technique requires integration over a single vibration period to compute body angular rates. An important and advantageous side effect of this integration process is the elimination of acceleration measurement bias and noise from the estimation process.

Vibratory angular rate sensors, usually called Coriolis sensors, are equipped with a vibrating element to measure angular rates, rather than a rotating mass in conventional gyroscopes. Coriolis sensors are also rugged like accelerometers. Basic operation of these single-axis angular rate sensors is detailed by Barnaby and Chatterton,[8] Quick,[9] Newton,[10] and Burdess and Wren.[11] More recently, Gallacher et al.[12] developed the theory of operation for an integrated three-axis vibrating gyroscope. Whether single-axis or multiple-axis devices, vibrating gyroscopes create an estimate of angular rates based on the dynamic behavior of a system with vibrating elements.

This Note presents a new concept for estimating angular velocity and angular acceleration of a body using four fixed triaxial acceleration measurements and three vibrating triaxial acceleration measurements. The technique has the advantage of not requiring integration while still properly resolving the algebraic sign of the angular rates. Computation of angular rates is based strictly on kinematic analysis of rigid bodies, which is in contrast to vibrating gyroscopes.

System Geometry

For convenience, the analysis to follow assumes the sensors are arranged in a specific manner. Consider Fig. 1, which shows a total of seven triaxial acceleration sensors mounted to a rigid body. The four sensors shown as small cubes, F_0, F_1, F_2, and F_3, are fixed to the body. The other three sensors, V_1, V_2, and V_3, shown as spheres

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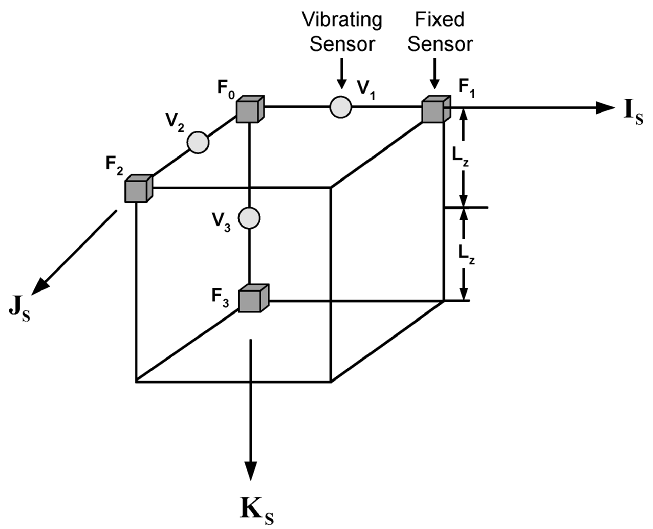

Fig. 1 Sensor cube geometry.

are forced to oscillate linearly about the center of a line from the origin of the sensor frame, F_0, to the respective fixed acceleration sensor. This induces known motion into the system that may be exploited later. Sensors one, two, and three for both the fixed and vibrating cases lie along the $\boldsymbol{I}_s$, $\boldsymbol{J}_s$, and $\boldsymbol{K}_s$ axes, respectively. The sensor reference frame is fixed and aligned to the body of interest. The vibrating sensors are positioned with an offset bias of L_x, L_y, and L_z from the origin of the sensor reference frame and oscillate at a frequency of ω_x, ω_y, and ω_z, with an amplitude of n_x, n_y, and n_z, respectively.

Angular Rate Estimation Using Fixed and Vibrating Single-Axis Acceleration Sensors

The technique developed by Merhav[7] utilizes a set of three fixed and three vibrating single-axis acceleration measurements. With reference to Fig. 1, sensors F_1, F_2, F_3, V_1, V_2, and V_3 are employed. (Sensor F_0 at the origin is not required.) A key element of this technique is that the single-axis acceleration measurement of each vibrating sensor is perpendicular to its direction of motion. Therefore, it is mounted in such a way that the sensing axis of the accelerometer is perpendicular to the direction of vibrating motion. Acceleration measurements from sensors F_1 and V_1 are along the $\boldsymbol{J}_s$ axis, whereas the motion of sensor V_1 is along the $\boldsymbol{I}_s$ axis. In the same manner, acceleration measurements from sensors F_2 and V_2 are along the $\boldsymbol{K}_s$ axis, whereas the motion of sensor V_2 is along the $\boldsymbol{J}_s$ axis. Following this pattern, acceleration measurements from sensors F_3 and V_3 are along the $\boldsymbol{I}_s$ axis, whereas the motion of V_3 is along the $\boldsymbol{K}_s$ axis.

The acceleration of each vibrating sensor can be computed in terms of the fixed sensor acceleration using the general one-point-moving-on-a-rigid-body formula (Ref. 3)

$$\boldsymbol{a}_{V_j/I} = \boldsymbol{a}_{F_j/I} + \boldsymbol{a}_{V_j/B} + \boldsymbol{\alpha}_{B/I} \times \boldsymbol{r}_{F_j\to V_j} + \boldsymbol{\omega}_{B/I} \times \left(\boldsymbol{\omega}_{B/I} \times \boldsymbol{r}_{F_j\to V_j}\right) + 2\boldsymbol{\omega}_{B/I} \times \boldsymbol{v}_{V_j/B} \tag{1}$$

All terms in Eq. (1) are expressed in the sensor reference frame, as follows:

$$\boldsymbol{a}_{F_j/I} = a_x^{F_j}\boldsymbol{I}_s + a_y^{F_j}\boldsymbol{J}_s + a_z^{F_j}\boldsymbol{K}_s \tag{2}$$

$$\boldsymbol{a}_{V_j/I} = a_x^{V_j}\boldsymbol{I}_s + a_y^{V_j}\boldsymbol{J}_s + a_z^{V_j}\boldsymbol{K}_s \tag{3}$$

$$\boldsymbol{a}_{V_j/B} = \tilde{a}_x^{V_j}\boldsymbol{I}_s + \tilde{a}_y^{V_j}\boldsymbol{J}_s + \tilde{a}_z^{V_j}\boldsymbol{K}_s \tag{4}$$

$$\boldsymbol{\omega}_{B/I} = p\boldsymbol{I}_s + q\boldsymbol{J}_s + r\boldsymbol{K}_s \tag{5}$$

$$\boldsymbol{\alpha}_{B/I} = \dot{p}\boldsymbol{I}_s + \dot{q}\boldsymbol{J}_s + \dot{r}\boldsymbol{K}_s \tag{6}$$

$$\boldsymbol{r}_{F_j\to V_j} = \Delta x_{F_j\to V_j}\boldsymbol{I}_s + \Delta y_{F_j\to V_j}\boldsymbol{J}_s + \Delta z_{F_j\to V_j}\boldsymbol{K}_s \tag{7}$$

$$\boldsymbol{v}_{V_j/B} = \tilde{v}_x^{V_j}\boldsymbol{I}_s + \tilde{v}_y^{V_j}\boldsymbol{J}_s + \tilde{v}_z^{V_j}\boldsymbol{K}_s \tag{8}$$

The tilde used above vector components signifies the quantity is with respect to the body and not inertial space. The components of Eq. (1) in the sensor reference frame are provided by

$$\begin{Bmatrix} a_x^{V_j} \\ a_y^{V_j} \\ a_z^{V_j} \end{Bmatrix} = \begin{Bmatrix} a_x^{F_j} \\ a_y^{F_j} \\ a_z^{F_j} \end{Bmatrix} + \begin{Bmatrix} \tilde{a}_x^{V_j} \\ \tilde{a}_y^{V_j} \\ \tilde{a}_z^{V_j} \end{Bmatrix} + 2\begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}\begin{Bmatrix} \tilde{v}_x^{V_j} \\ \tilde{v}_y^{V_j} \\ \tilde{v}_z^{V_j} \end{Bmatrix} + \begin{bmatrix} -q^2-r^2 & pq-\dot{r} & pr+\dot{q} \\ pq+\dot{r} & -p^2-r^2 & qr-\dot{p} \\ pr-\dot{q} & qr+\dot{p} & -p^2-q^2 \end{bmatrix}\begin{Bmatrix} \Delta x_{F_j\to V_j} \\ \Delta y_{F_j\to V_j} \\ \Delta z_{F_j\to V_j} \end{Bmatrix} \tag{9}$$

Harmonic motion of each individual vibrating sensor is along a single axis. This is known motion. Thus, the components of the relative position, velocity, and acceleration vectors can be expressed as

$$\Delta x_{F_2\to V_2} = \Delta x_{F_3\to V_3} = 0 \tag{10}$$

$$\Delta y_{F_1\to V_1} = \Delta y_{F_3\to V_3} = 0 \tag{11}$$

$$\Delta z_{F_1\to V_1} = \Delta z_{F_2\to V_2} = 0 \tag{12}$$

$$\Delta x_{F_1\to V_1} = n_x \sin\omega_x t - L_x \tag{13}$$

$$\Delta y_{F_2\to V_2} = n_y \sin\omega_y t - L_y \tag{14}$$

$$\Delta z_{F_3\to V_3} = n_z \sin\omega_z t - L_z \tag{15}$$

$$\tilde{v}_x^{V_2} = \tilde{v}_x^{V_3} = \tilde{v}_y^{V_1} = \tilde{v}_y^{V_3} = \tilde{v}_z^{V_1} = \tilde{v}_z^{V_2} = 0 \tag{16}$$

$$\tilde{v}_x^{V_1} = n_x\omega_x \cos\omega_x t \tag{17}$$

$$\tilde{v}_y^{V_2} = n_y\omega_y \cos\omega_y t \tag{18}$$

$$\tilde{v}_z^{V_3} = n_z\omega_z \cos\omega_z t \tag{19}$$

$$\tilde{a}_x^{V_2} = \tilde{a}_x^{V_3} = \tilde{a}_y^{V_1} = \tilde{a}_y^{V_3} = \tilde{a}_z^{V_1} = \tilde{a}_z^{V_2} = 0 \tag{20}$$

$$\tilde{a}_x^{V_1} = -n_x\omega_x^2 \sin\omega_x t \tag{21}$$

$$\tilde{a}_y^{V_2} = -n_y\omega_y^2 \sin\omega_y t \tag{22}$$

$$\tilde{a}_z^{V_3} = -n_z\omega_z^2 \sin\omega_z t \tag{23}$$

Note that the vibrating sensors are positioned with an offset bias of L_x, L_y, or L_z from the origin of the sensor reference frame. The time derivative of this offset bias is zero due to the assumption that the sensor cube behaves as a rigid body. Extracting the $\boldsymbol{I}_s$ axis component of Eq. (9) applied to sensors F_3 and V_3, the $\boldsymbol{J}_s$ axis component applied to sensors F_1 and V_1, and the $\boldsymbol{K}_s$ axis components applied to sensors F_2 and V_2 yields

$$\begin{Bmatrix} a_x^{V_3} \\ a_y^{V_1} \\ a_z^{V_2} \end{Bmatrix} - \begin{Bmatrix} a_x^{F_3} \\ a_y^{F_1} \\ a_z^{F_2} \end{Bmatrix} = \begin{Bmatrix} (pr+\dot{q})(n_z\sin\omega_z t - L_z) \\ (pq+\dot{r})(n_x\sin\omega_x t - L_x) \\ (qr+\dot{p})(n_y\sin\omega_y t - L_y) \end{Bmatrix} + \begin{Bmatrix} 2qn_z\omega_z\cos\omega_z t \\ 2rn_x\omega_x\cos\omega_x t \\ 2pn_y\omega_y\cos\omega_y t \end{Bmatrix} \tag{24}$$

If the sensors are forced to oscillate at a sufficiently high frequency, the angular velocity may be assumed constant over a single cycle of sensor vibration. In this case, the angular acceleration terms go to zero. Multiplying the components of Eq. (24) by $\cos\omega_z t$, $\cos\omega_x t$ and $\cos\omega_y t$, respectively, and integrating over a single cycle of sensor vibration isolates the angular velocity components:

$$p = \frac{1}{2\pi n_y}\int_0^{2\pi/\omega_y} \cos\omega_y t\left(a_z^{V_2} - a_z^{F_2}\right)\mathrm{d}t \tag{25}$$

$$q = \frac{1}{2\pi n_z}\int_0^{2\pi/\omega_z} \cos\omega_z t\left(a_x^{V_3} - a_x^{F_3}\right)\mathrm{d}t \tag{26}$$

$$r = \frac{1}{2\pi n_x}\int_0^{2\pi/\omega_x} \cos\omega_x t\left(a_y^{V_1} - a_y^{F_1}\right)\mathrm{d}t \tag{27}$$

Thus, to estimate the angular rate components of a body in the sensor reference frame, three quadratures must be performed using the fixed and vibrating sensor measurements. Note that the integrals are closely related to the first cosine wave harmonic amplitude. The difference between the fixed and vibrating accelerations form the multiplicative constant applied to the cosine wave. This technique has the significant advantage that it naturally eliminates sensor bias and noise from the estimates of the angular rate components. To see this, note that sensor bias adds a constant error to the acceleration differences in Eqs. (25–27), which, when multiplied by $\cos\omega_x t$, $\cos\omega_y t$, or $\cos\omega_z t$ and integrated over a single cycle of sensor vibration, is zero. Sensor noise adds a zero mean random quantity to the acceleration differences in Eqs. (25–27). Sensor noise, when multiplied by a sinusoid and integrated over a single cycle also yields zero. The disadvantage of this technique is that numerical quadrature must be accurately computed to estimate the angular velocity components. Also, the algorithm was developed based on the assumption that angular rates are constant over a cycle of the vibrating sensors, thus, limiting the useful frequency range of measurement.

Angular Rate Estimation Using Fixed Triaxial Acceleration Sensors

With reference to Fig. 1, the technique developed by Costello and Jitpraphai[6] uses four fixed triaxial acceleration sensors. The vibrating acceleration sensors are disregarded. Application of Eq. (1) to sensor point combinations $F_0 - F_1$, $F_1 - F_2$, and $F_2 - F_3$ generates three sets of equations that are concatenated into matrix form as

$$A = MR \tag{28}$$

$$A = \begin{bmatrix} a_x^{F_0} - a_x^{F_1} & a_x^{F_1} - a_x^{F_2} & a_x^{F_2} - a_x^{F_3} \\ a_y^{F_0} - a_y^{F_1} & a_y^{F_1} - a_y^{F_2} & a_y^{F_2} - a_y^{F_3} \\ a_z^{F_0} - a_z^{F_1} & a_z^{F_1} - a_z^{F_2} & a_z^{F_2} - a_z^{F_3} \end{bmatrix} \tag{29}$$

$$M = \begin{bmatrix} -q^2 - r^2 & -\dot r + pq & \dot q + pr \\ \dot r + pq & -p^2 - r^2 & -\dot p + qr \\ -\dot q + pr & \dot p + qr & -p^2 - q^2 \end{bmatrix} \tag{30}$$

$$R = 2\begin{bmatrix} -L_x & L_x & 0 \\ 0 & -L_y & L_y \\ 0 & 0 & -L_z \end{bmatrix} \tag{31}$$

The matrix A is populated by sensor measurements and is known at each discrete time instant. The distance matrix R is defined by sensor geometry, and the matrix M contains the unknown quantities that are to be estimated. Provided the distance matrix R is nonsingular, the matrix M can be computed. Previous efforts have yielded algorithms to compute angular rate components using the elements of the M matrix. See Refs. 4–6 for details on these algorithms. The results obtained are given as follows:

$$p = s_p\left\{\tfrac{1}{2}\left[(M_{1,1} - M_{2,2})^2 + (M_{1,2} + M_{2,1})^2\right]^{\frac{1}{2}} + \tfrac{1}{2}(M_{1,1} - M_{2,2})\right\}^{\frac{1}{2}} \tag{32}$$

$$q = s_q\left\{\tfrac{1}{2}\left[(M_{1,1} - M_{2,2})^2 + (M_{1,2} + M_{2,1})^2\right]^{\frac{1}{2}} - \tfrac{1}{2}(M_{1,1} - M_{2,2})\right\}^{\frac{1}{2}} \tag{33}$$

$$r = s_r\left\{\tfrac{1}{2}\left[(M_{1,1} - M_{2,2})^2 + (M_{1,2} + M_{2,1})^2\right]^{\frac{1}{2}} - \tfrac{1}{2}(M_{1,1} + M_{2,2})\right\}^{\frac{1}{2}} \tag{34}$$

where s_p, s_q, and s_r are the algebraic signs of p, q, and r. Angular rate estimation using only fixed triaxial acceleration measurement is sensitive to acceleration measurement noise, particularly for some critical combinations of sensor geometry and angular rates. Moreover, the algebraic sign of the angular rates cannot be ascertained using only fixed triaxial acceleration measurement because two valid solutions exist.

Angular Rate Estimation Using Fixed and Vibrating Triaxial Acceleration Sensors

Again consult Fig. 1: Consider estimating angular rates using fixed and vibrating triaxial acceleration sensors. The vibrating sensors create known motion. Thus, the velocity and acceleration of these sensors relative to the sensor reference frame are known. When this information is utilized, the angular velocity components may be computed. Applying Eq. (1) to each fixed and rotating point combination yields

$$\tilde A = M\tilde R + 2S\tilde V \tag{35}$$

$$\tilde A = \begin{bmatrix} a_x^{V_1} - a_x^{F_1} - \tilde a_x^{V_1} & a_x^{V_2} - a_x^{F_2} - \tilde a_x^{V_2} & a_x^{V_3} - a_x^{F_3} - \tilde a_x^{V_3} \\ a_y^{V_1} - a_y^{F_1} - \tilde a_y^{V_1} & a_y^{V_2} - a_y^{F_2} - \tilde a_y^{V_2} & a_y^{V_3} - a_y^{F_3} - \tilde a_y^{V_3} \\ a_z^{V_1} - a_z^{F_1} - \tilde a_z^{V_1} & a_z^{V_2} - a_z^{F_2} - \tilde a_z^{V_2} & a_z^{V_3} - a_z^{F_3} - \tilde a_z^{V_3} \end{bmatrix} \tag{36}$$

$$\tilde R = \begin{bmatrix} n_x\sin\omega_x t - L_x & 0 & 0 \\ 0 & n_y\sin\omega_y t - L_y & 0 \\ 0 & 0 & n_z\sin\omega_z t - L_z \end{bmatrix} \tag{37}$$

$$\tilde V = \begin{bmatrix} n_x\omega_x\cos\omega_x t & 0 & 0 \\ 0 & n_y\omega_y\cos\omega_y t & 0 \\ 0 & 0 & n_z\omega_z\cos\omega_z t \end{bmatrix} \tag{38}$$

$$S = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \tag{39}$$

The matrix M is given by Eq. (30). Solving for S yields

$$S = \tfrac{1}{2}(\tilde A - AR^{-1}\tilde R)\tilde V^{-1} \tag{40}$$

which is utilized to compute the components of the angular rate vector. Solving for angular rates yields

$$p = \frac{\left(a_z^{F_2} - a_z^{F_0}\right)(L_y - n_y\sin\omega_y t) + 2L_y\left(a_z^{V_2} - a_z^{F_2}\right)}{8L_y n_y\omega_y\cos\omega_y t} - \frac{\left(a_y^{F_3} - a_y^{F_0}\right)(L_z - n_z\sin\omega_z t) + 2L_z\left(a_y^{V_3} - a_y^{F_3}\right)}{8L_z n_z\omega_z\cos\omega_z t} \tag{41}$$

$$q = \frac{\left(a_x^{F_3} - a_x^{F_0}\right)(L_z - n_z\sin\omega_z t) + 2L_z\left(a_x^{V_3} - a_x^{F_3}\right)}{8L_z n_z\omega_z\cos\omega_z t} - \frac{\left(a_z^{F_1} - a_z^{F_0}\right)(L_x - n_x\sin\omega_x t) + 2L_x\left(a_z^{V_1} - a_z^{F_1}\right)}{8L_x n_x\omega_x\cos\omega_x t} \tag{42}$$

$$r = \frac{\left(a_y^{F_1} - a_y^{F_0}\right)(L_x - n_x \sin\omega_x t) + 2L_x\left(a_y^{V_1} - a_y^{F_1}\right)}{8L_x n_x \omega_x \cos\omega_x t} - \frac{\left(a_x^{F_2} - a_x^{F_0}\right)(L_y - n_y \sin\omega_y t) + 2L_y\left(a_x^{V_2} - a_x^{F_2}\right)}{8L_y n_y \omega_y \cos\omega_y t} \tag{43}$$

In the computation of the angular rates p, q, and r, a problem arises as the combination of ωt approaches an odd multiple of $\pi/2$, where $\cos\omega t = 0$ and the solutions listed in Eqs. (41–43) become singular. This problem is overcome by sampling the data when the vibrating sensor is near the middle of its vibration cycle. In this way, it is ensured that $\cos\omega t \approx 1$. This requires knowledge of the oscillation frequency and the sampling rate of data collection. When proper sampling is used where $\sin\omega t = 0$ and $\cos\omega t = 1$, angular rates estimates are

$$p = \frac{2a_z^{V_2} - a_z^{F_2} - a_z^{F_0}}{8n_y\omega_y} - \frac{2a_y^{V_3} - a_y^{F_3} - a_y^{F_0}}{8n_z\omega_z} \tag{44}$$

$$q = \frac{2a_x^{V_3} - a_x^{F_3} - a_x^{F_0}}{8n_z\omega_z} - \frac{2a_z^{V_1} - a_z^{F_1} - a_z^{F_0}}{8n_x\omega_x} \tag{45}$$

$$r = \frac{2a_y^{V_1} - a_y^{F_1} - a_y^{F_0}}{8n_x\omega_x} - \frac{2a_x^{V_2} - a_x^{F_2} - a_x^{F_0}}{8n_y\omega_y} \tag{46}$$

When the angular rates from Eqs. (41–43), the matrix $M = AR^{-1}$ given from acceleration measurements along with sensor geometry, and the known structure of the matrix M given by Eq. (30) are used, the angular acceleration components may be determined as well:

$$\dot{p} = \frac{a_z^{F_2} - a_z^{F_0}}{4L_y} - \frac{a_y^{F_3} - a_y^{F_0}}{4L_z} \tag{47}$$

$$\dot{q} = \frac{a_x^{F_3} - a_x^{F_0}}{4L_z} - \frac{a_z^{F_1} - a_z^{F_0}}{4L_x} \tag{48}$$

$$\dot{r} = \frac{a_y^{F_1} - a_y^{F_0}}{4L_x} - \frac{a_x^{F_2} - a_x^{F_0}}{4L_y} \tag{49}$$

Note that, in comparison to the method developed by Merhav,[7] the method described here requires more sensors, but does not require integration over a cycle of vibration. Furthermore, it makes no assumptions about the angular rates as constant over a single cycle of sensor vibration, which extends the usable frequency band of the method.

Error Analysis

When Eqs. (41–46) are used, angular velocity and angular acceleration components may be computed. To highlight the effect of errors in physical parameters on the estimation of angular rates, sensitivity analysis is employed. Outputs of the estimation algorithm (angular velocity and acceleration components) are placed in the vector $\boldsymbol{Y} = [p \;\; q \;\; r \;\; \dot{p} \;\; \dot{q} \;\; \dot{r}]$, whereas sensor configuration design parameters (sensor cube dimensions, amplitude of vibration of the vibrating sensors, and frequency of oscillation) are placed in the vector $\boldsymbol{U} = [L_x \;\; L_y \;\; L_z \;\; n_x \;\; n_y \;\; n_z \;\; \omega_x \;\; \omega_y \;\; \omega_z]$. The estimation algorithm maps $\boldsymbol{U}$ to $\boldsymbol{Y}$ through a nonlinear operator F:

$$\boldsymbol{Y} = F(\boldsymbol{U}) \tag{50}$$

Equation (50) may be approximated locally using a Taylor series retaining only the linear term. This provides a relationship between errors in the input parameters and errors in the estimates:

$$\delta\boldsymbol{Y} = G\delta\boldsymbol{U} \tag{51}$$

The matrix G represents the sensitivity of the computed angular velocity and acceleration components to errors in inputs. Because estimation of $\dot{p}$, $\dot{q}$, and $\dot{r}$ is based only on the fixed sensors, these estimates are not sensitive to errors in vibrating sensor amplitude or frequency. Also, estimation of p does not involve L_x, n_x, or ω_x, so that inaccuracies in these parameters do not alter the estimate of p. In a similar manner, estimation of q is insensitive to errors in L_y, n_y, and ω_y, whereas estimation of r is insensitive to errors in L_z, n_z, and ω_z. Differentiating the angular rate and angular acceleration components relative to L_x, L_y, L_z, n_x, n_y, n_z, ω_x, ω_y, and ω_z yields the elements of the error sensitivity matrix. After proper sampling is assumed, the nonzero elements of the error sensitivity matrix are

$$G_{15} = \frac{\partial p}{\partial n_y} = \frac{a_z^{F_2} + a_z^{F_0} - 2a_z^{V_2}}{8n_y^2\omega_y} \tag{52}$$

$$G_{16} = \frac{\partial p}{\partial n_z} = -\frac{a_y^{F_3} + a_y^{F_0} - 2a_y^{V_3}}{8n_z^2\omega_z} \tag{53}$$

$$G_{18} = \frac{\partial p}{\partial \omega_y} = \frac{a_z^{F_0} + a_z^{F_2} - 2a_z^{V_2}}{8n_y\omega_y^2} + \frac{\left(a_z^{F_0} - a_z^{F_2}\right)t}{8L_y\omega_y} \tag{54}$$

$$G_{19} = \frac{\partial p}{\partial \omega_z} = \frac{a_y^{F_0} + a_y^{F_3} - 2a_y^{V_3}}{8n_z\omega_z^2} - \frac{\left(a_y^{F_0} - a_y^{F_3}\right)t}{8L_z\omega_z} \tag{55}$$

$$G_{24} = \frac{\partial q}{\partial n_x} = \frac{2a_z^{V_1} - a_z^{F_0} - a_z^{F_1}}{8n_x^2\omega_x} \tag{56}$$

$$G_{26} = \frac{\partial q}{\partial n_z} = \frac{a_x^{F_0} + a_x^{F_3} - 2a_x^{V_3}}{8n_z^2\omega_z} \tag{57}$$

$$G_{27} = \frac{\partial q}{\partial \omega_x} = -\frac{a_z^{F_0} + a_z^{F_1} - 2a_z^{V_1}}{8n_x\omega_x^2} - \frac{\left(a_z^{F_0} - a_z^{F_1}\right)t}{8L_x\omega_x} \tag{58}$$

$$G_{29} = \frac{\partial q}{\partial \omega_z} = -\frac{a_x^{F_0} + a_x^{F_3} - 2a_x^{V_3}}{8n_z\omega_z^2} + \frac{\left(a_x^{F_0} - a_z^{F_3}\right)t}{8L_z\omega_z} \tag{59}$$

$$G_{34} = \frac{\partial r}{\partial n_x} = \frac{a_y^{F_0} + a_y^{F_1} - 2a_y^{V_1}}{8n_x\omega_x^2} \tag{60}$$

$$G_{35} = \frac{\partial r}{\partial n_y} = -\frac{a_x^{F_0} - a_x^{F_2} - 2a_x^{V_2}}{8n_y^2\omega_y} \tag{61}$$

$$G_{37} = \frac{\partial r}{\partial \omega_x} = -\frac{a_y^{F_0} + a_y^{F_1} - 2a_y^{V_1}}{8n_x\omega_x^2} + \frac{\left(a_y^{F_0} - a_y^{F_1}\right)t}{8L_x\omega_x} \tag{62}$$

$$G_{39} = \frac{\partial r}{\partial \omega_y} = -\frac{a_x^{F_0} + a_x^{F_2} - 2a_x^{V_2}}{8n_y\omega_y^2} - \frac{\left(a_x^{F_0} - a_x^{F_2}\right)t}{8L_y\omega_y} \tag{63}$$

$$G_{42} = \frac{\partial \dot{p}}{\partial L_y} = \frac{a_z^{F_0} - a_z^{F_2}}{4L_y^2} \tag{64}$$

$$G_{43} = \frac{\partial \dot{p}}{\partial L_z} = -\frac{a_y^{F_0} - a_y^{F_3}}{4L_z^2} \tag{65}$$

$$G_{51} = \frac{\partial \dot{q}}{\partial L_x} = -\frac{a_z^{F_0} - a_z^{F_1}}{4L_x^2} \tag{66}$$

$$G_{53} = \frac{\partial \dot{q}}{\partial L_z} = \frac{a_x^{F_0} - a_x^{F_3}}{4L_z^2} \tag{67}$$

$$G_{61} = \frac{\partial \dot{r}}{\partial L_x} = \frac{a_y^{F_0} - a_y^{F_1}}{4L_x^2} \tag{68}$$

$$G_{62} = \frac{\partial \dot{r}}{\partial L_y} = -\frac{a_x^{F_0} - a_x^{F_2}}{4L_y^2} \tag{69}$$

Notice that the acceleration measurements enter the estimation equations and error sensitivity matrix elements in a linear fashion. Thus, the relative error in acceleration measurements causes the same relative error in angular rate and angular acceleration estimates. Also, if clusters of these sensors are used, the effect of accelerometer noise, bias, scale factor, and cross axis sensitivity can be steadily reduced through averaging. Also notice that the design parameters of the sensor system all appear in the denominator of elements of the error sensitivity matrix. Thus, to reduce sensitivity of the sensor system to errors in sensor system geometry, vibrating sensor amplitude, and vibrating sensor frequency, the design parameters L_x, L_y, L_z, n_x, n_y, n_z, ω_x, ω_y, and ω_z should be set as large as practically allowable. The error sensitivity of angular rates to vibrating sensor frequency (G_{18}, G_{19}, G_{27}, G_{29}, G_{37}, and G_{38}) is time dependent. Thus, a self-regulating measurement of the vibrating sensor frequency is required for practical implementation to avoid drift in the angular rate estimates.

Conclusions

A method for estimating angular rates and angular accelerations of a body using clusters of seven triaxial linear acceleration measurements is presented. The method employs four triaxial accelerometers fixed to the body and three triaxial accelerometers that vibrate at a known frequency with respect to the body. Although other solutions to this problem exist, the method reported here is unique in that it does not require integration and also properly resolves the algebraic sign of the angular rates. To achieve minimum sensitivity to errors in the geometric arrangement of the sensor suite and the vibrating sensor amplitude and frequency, the design parameters should be maximized. In practical implementation, the vibrating sensor frequencies must be measured to avoid drift in the angular rate estimates.

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